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ON A THEOREM OF KEMER

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ABSTRACT

Kemer has shown that the standard identity s_n implies some Capelli identity d_m. We prove that m is bounded above by a subexponential function of n.

In [1] Kemer shows that every (associative, characteristic zero) algebra which satisfies a standard identity s_n must also satisfy some Capelli identity d_m. Let

$k(n)$ = the least integer such that $d_{k(n)}$ is a consequence of s_n.

The function $k(n)$ is not discussed explicitly in [1], although it is not difficult to show from the proof of Theorem 1 that, for n even,

$$k(n) \leqq \left(\left(\frac{n}{2}\right)^2 + 1\right)^n.$$

In this paper we show that $k(n)$ is a subexponential function: if m = the least integer $\geqq \log_2(3n-1)$, then

$$k(n) \leqq \left(\left[\frac{n}{2}\right]^2 + 1\right)^m.$$

Our proof involves only a small modification of Kemer's.

Since $m \times m$ matrices satisfy s_{2m} but not d_{m^2}, $k(2m) \geqq m^2 + 1$. We do not know whether $k(n)$ is bounded above by any polynomial.

We keep the notation of [1]:

NOTATION. (a) $S_n = \Sigma_{\sigma \in S_n} (-1)^\sigma x_{\sigma 1} \cdots x_{\sigma n}$. I_n = the T-ideal generated by s_n.

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(b) $d_m(x_1,\dots,x_m;\ y_1,\dots,y_{m-1}) = \sum_{\sigma\in S_m}(-1)^\sigma x_{\sigma 1}y_1x_{\sigma 2}y_2\cdots y_{m-1}x_{\sigma m}$. V_m = the T-ideal generated by d_m together with all polynomials gotten from d_m by formally setting some subset of the y's equal to 1.

(c) G = the F-algebra generated by $y_1, y_2, \dots, e_1, e_2, \dots$ with relations $y_iuy_j + y_juy_i = 0$ for all u.

(d) W_m = the set of words of length m in $e_1, e_2, \dots$. If $w_1,\dots,w_n \in W_m$, $g_1,\dots,g_{n-1}\in G$ and $\sigma\in S_n$, then

$$w_{\sigma 1}g_1w_{\sigma 2}g_2\cdots g_{n-1}w_{\sigma n} = \varepsilon w_1g_1w_2g_2\cdots g_{n-1}w_{n-1},$$

where $\varepsilon = 1$ if m is even and $\varepsilon = (-1)^\sigma$ if m is odd.

(e) T_m = the ideal of G generated by W_m.

(f) If A is a T-ideal, then $A(G)$ is the set of evaluations of A on G. Note that W_m and hence T_m is contained in $I_m(G)$.

We also recall

LEMMA (Kemer). *Let A be a T-ideal. Then A contains V_m if and only if $T_1^m \subseteq A(G)$.*

Following Kemer we take $n = 2r$ to be even. Now the proof of his theorem 1 can be modfied to yield:

LEMMA. *Let $h_r(x_1,\dots,x_r)$ be the polynomial $\sum_{\sigma\in S_r}x_{\sigma 1}\cdots x_{\sigma r}$, and let $x_1,\dots,x_{r+1}\in W_mG$.*

(1) *If m is odd, then*

$$h_r(x_1,\dots,x_r)x_{r+1}\in T_{2m} + I_n(G).$$

(2) *If m is even, then*

$$h_r(x_1,\dots,x_r)x_{r+1}\in T_{2m-1} + I_n(G).$$

PROOF. (1) Let $x_i = w_ig_i$, $w_i\in W_m$, $i = 1,\dots,r+1$, and calculate

$$h_r(x_1,\dots,x_r)x_{r+1} = \sum_{\sigma\in S_r} w_{\sigma 1}g_{\sigma 1}\cdots w_{\sigma r}g_{\sigma r}w_{r+1}g_{r+1}$$

$$= \sum_{\sigma\in S_r}(-1)^\sigma w_1g_{\sigma 1}\cdots w_rg_{\sigma r}w_{r+1}g_{r+1}$$

$$= \sum_{\sigma,\tau\in S_r}(-1)^\sigma(-1)^\tau w_{\tau 1}g_{\sigma 1}\cdots w_{\tau r}g_{\sigma r}w_{r+1}g_{r+1}$$

$$= \frac{1}{r!}s_n(w_1, g_1,\dots,w_r, g_r)w_{r+1}g_{r+1} + \text{an element of } T_{2m}.$$

(2) Let $x_i = w_i g_i$, where $w_i \in W_{m-1}$, $g_i \in T_1 G$, $i = 1, \dots, r+1$ and proceed as in (1).

REMARK. If $n = 2r+1$ is odd the lemma is still true: one need only consider $s_n(w_1, g_1, \dots, w_r, g_r, w_{r+1})g_{r+1}$ instead of $s_n(w_1, g_1, \dots, w_r, g_r)w_{r+1}g_{r+1}$.

COROLLARY. *If m is even, then $T_m^{r^2+1} \subseteq T_{2m} + I_n(G)$ and if m is odd then $T_m^{r^2+1} \subseteq T_{2m-1} + I_n(G)$.*

PROOF. By Razmyslov's version of the Nagata–Higman theorem [2], if an algebra satisfies $h_r(x_1, \dots, x_r)$ then it must be nilpotent of index r^2. So

$$x_1 \cdots x_{r^2} = \text{linear combination of terms of the form } u_1 h_r(u_2, \dots, u_{r+1})u_{r+2},$$

where the u's are (possibly empty) words in the x's. Multiplying both sides on the right by x_{r^2+1} shows that if an algebra satisfies $h_r(x_1, \dots, x_r)x_{r+1}$ then it is nilpotent of index r^2+1.

Applying this to the previous lemma yields that $(W_m G)^{r^2+1}$ is contained in the appropriate ideal, and so T_m also is.

In light of this corollary we define

DEFINITION. Let f be the integer function given by $f(1) = 1$ and

$$f(m+1) = \begin{cases} 2f(m) & \text{if } m \text{ is odd,} \\ 2f(m) - 1 & \text{if } m \text{ is even.} \end{cases}$$

It is easy to show by induction that $T_1^{(r^2+1)^m} \subseteq T_{f(m)} + I_n(G)$. Since $T_n \subseteq I_n$, if $f(m) \geqq n$ then $T_1^{(r^2+1)^m} \subseteq I_n(G)$ or

PROPOSITION. *If $f(m) \geqq n$ and $k = (r^2+1)^m$, then $T_1^k \subseteq I_n$, i.e., s_n implies $d_{(r^2+1)^m}$.*

To calculate $f(m)$ we first remark that

$$f(m+2) = \begin{cases} 4f(m) - 2 & \text{if } m \text{ is even,} \\ 4f(m) - 1 & \text{if } m \text{ is odd.} \end{cases}$$

It is now a straightforward calculation using the theory of linear recursion that

$$f(m) = \begin{cases} \frac{1}{3}(2^m + 2) & m \text{ even,} \\ \frac{1}{3}(2^m + 1) & m \text{ odd.} \end{cases}$$

THEOREM. *Let m be an integer such that $m \geqq \log_2(2n-1)$ and $k=(r^2+1)^m$. Then s_n implies the Capelli identity d_k.*

REMARK. It is not known whether s_n implies d_m in arbitrary characteristic. Our proof will hold in a slightly weaker form in characteristic $\geqq n!$. The Nagata-Higman theorem is known for characteristic $\geqq n!$, but it is not clear whether Razmyslov's bound of n^2 will still hold in this case.

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